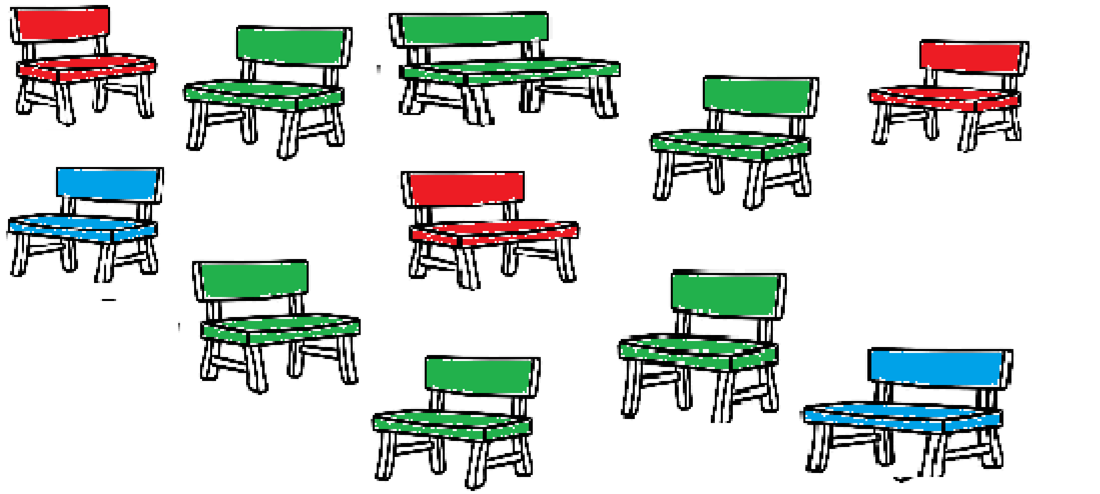
EIDMA

Lecture 10

- Stars-and-bars principle
- Properties of $\binom{n}{k}$

Recall the example:

In how many ways can one paint 10 benches with three colors, c_1, c_2 and c_3 (say red, green and blue) so that each bench is painted in one color? The order of colors is arbitrary but fixed.



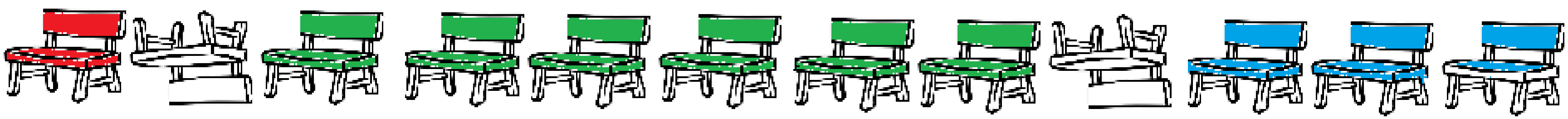
We solved the task in the case where benches and colors are distinguishable.
If colors are not distinguishable (we are color-blind), the question becomes identical with the question of painting the benches with just one color; benches distinguishable or not there is just one possibility.
This only leaves open the case where colors are distinguishable, and benches are not. In this case the question is not WHICH bench is painted what color but HOW MANY benches are painted red, how many are painted green and how many are painted blue.

Denote by x_i the number of benches painted c_i. Since each bench must be painted, $x_1 + x_2 + x_3 = 10$ and each x_i must be a nonnegative integer; $x_i = 0$ means no bench is painted c_i . This means, that we are looking for the number of nonnegative, integer solutions to the equation $x_1 + x_2 + x_3 = 10$.
Notice that we are after the *number of solutions* to this equation, not after solutions.

This problem is also called *counting 10-element "subsets with repetitions" (or combinations with repetitions) of a 3-element set* $\{c_1, c_2, c_3\}$. Each such "subset" is represented by an ordered triple of nonnegative integers, (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 10$ (meaning we take x_1 copies of c_1 etc.).
For example, (2,5,3) represents the (2+5+3) 10-element subset $\{c_1, c_1, c_2, c_2, c_2, c_2, c_2, c_3, c_3, c_3\}$ of $\{c_1, c_2, c_3\}$ with c_1 repeated 2 times, c_2 appearing 5 times and c_3 – 3 times. This is the same as $\{c_2, c_3, c_2, c_2, c_1, c_2, c_3, c_1, c_3, c_2\}$ but (5,3,2) denotes a "set" with 5 c_1's, 3 c_2's and 2 c_3's.
Another way of looking at this problem is this: in how many ways we can clone 3 sheep, Dolly, Molly and Polly so that we get 10 sheep, e.g., 2 Dollies, 5 Mollies and 3 Pollies.

A way of solving at the problem is to line-up the benches and split the line into three parts (some of them possibly empty). The first group will be painted c_1 the second c_2 and the third c_3. This can be achieved by inserting two separators into our line of benches.

A mental exercise: line-up 12 benches instead of 10 and tip over two of them – they will serve as the separators. In how many ways can you do this? You can line them up in one way only (because they are indistinguishable) and you can choose your two separators in $\binom{10+2}{2} = \binom{12}{2}$ = 66 ways.



Comprehension.

Show that the following 4 questions have the same answer:

1. We draw ten times from a box of 3 different objects, after each draw returning the drawn object to the box. What is the number of possible results (we disregard the order in which the objects are drawn)?
2. What is the number of polynomials of degree 10 with the coefficient of x^{10} equal to 1 and with 10 roots, all in $\{3,-1,4\}$?
3. What is the number of nonnegative integer solutions to $x_1+x_2+x_3=10$?
4. What is the number of 10-subsets with repetitions of a 3-element set?

Theorem.

The number of k-element subsets with unspecified number of repetitions of an n-element set is $\binom{k+n-1}{n-1}$.

Proof.

This is an easy generalization of the previous example.

This method is also known as "stars and bars". You split the row of k stars into n parts inserting $n-1$ vertical bars. For example, **|*****|*** represents the (2,5,3) 10-element subset with repetitions (2 red benches, 5 green and 3 blue ones) of {red, green, blue} and ||*********** - the (0,0,10) subset (all benches painted blue or we clone Polly 10 times).

Places for the separating bars can be chosen in $\binom{k+n-1}{n-1}$ ways. QED

Example.

What is the number of nondecreasing sequences of length k made of numbers from $\{1,2,\ldots,n\}$.

Solution.

Each such sequence begins with p_1 (possibly 0) 1-s, followed by p_2 2-s and so on. Which is a textbook example of a k-element subset with unlimited repetitions of the n-element set $\{1,2,\ldots,n\}$. Hence, the answer is $\binom{k+n-1}{n-1}$.

Notice that the condition "nondecreasing" has no effect on the answer.

Example.

What is the number of nondecreasing sequences of length k made of numbers from $\{1,2,\ldots,n\}$ and containing at least one of each of the numbers.

Solution.

Each such sequence begins with p_1 ($p_1 \geq 1$) 1-s, followed by p_2 ($p_2 \geq 1$) 2-s and so on. This leaves exactly $k - n$ places in the sequence to be occupied by those "extra" numbers and it means also that the answer in the case $k < n$ is nil. In other words, we must calculate the number of $k - n$ - element subsets with repetitions of the n-element set $\{1,2,\ldots,n\}$, which is $\binom{(k-n)+n-1}{n-1} = \binom{k-1}{n-1}$.

$$\underbrace{(1,\ldots,2,\ldots,\ldots\ldots,n-1,\ldots,n,\ldots)}_{\text{length } k \text{ but } n \text{ places taken}}$$

Let's see what it means in practice. First, take $k = n$. This means we want to count n-long nondecreasing sequences of $1,2, \ldots, n$. Clearly, there is only one, $(1,2, \ldots, n)$. This supports our formula because $\binom{n-1}{n-1} = 1$. Let $k = n + 1$. Our sequences look like $(1,1,2, \ldots, n)$, $(1,2,2,3, \ldots, n)$ etc. which means the answer is n and $\binom{n+1-1}{n-1} = \binom{n}{n-1} = n$.

Comprehension.

What happens if you reverse the meaning of stars and bars?

The number of such sequences is of course the same as before, you choose places for $n-1$ stars in the $k+n-1$ -long sequence instead of $n-1$ bars, but what is the meaning of this result, say in the language of benches and colors? Or in terms of sets and subsets with repetitions? Or in terms of counting solutions of an equation?

PROPERTIES OF THE NEWTON'S BINOMIAL COEFFICIENT $\binom{n}{k}$

Why the hell Newton's?
Why the hell is it called binomial?
Why the hell is it called a coefficient?

I quote Wikipedia:

Andreas von Ettingshausen introduced the notation $\binom{n}{k}$ in 1826, although the numbers were known centuries earlier. The earliest known detailed discussion of binomial coefficients is in a tenth-century commentary, by Halayudha, on an ancient Sanskrit text, Pingala's *Chandaḥśāstra*. In about 1150, the Indian mathematician Bhaskaracharya gave an exposition of binomial coefficients in his book *Līlāvatī*.

Theorem. (Newton's binomial theorem)
For every two numbers a, b

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof 1. (no induction, some combinatorics!)
$(a+b)^n = (a+b)(a+b) \dots (a+b) = p_1 p_2 \dots p_n$
Opening parenthesis, we get the sum of n-long products of a's and b's. How many of these products have exactly k a's? As many as ways of choosing k out of n pairs $p_1, p_2, \dots, p_n$. Each of the chosen k pairs contributes an a, the remaining $n-k$ pairs contribute b's to the product.
The number of choices is $\binom{n}{k}$. That's why "coefficient.

A lesson to remember: we can use combinatorics in algebra.

Theorem.

For every two natural numbers *n*,*k*, $k \leq n$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

In other words, *the number of k-element subset in an n-element set is the combined number of k- and* $(k-1)$*-element subsets in an* $(n-1)$*-element set.*

Proof 1. (no induction, pure algebra. Boring but works!)

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{(n-k-1)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!} = (\text{algebra})$$

$$\frac{(n-1)!(n-k)}{(n-k)(n-k-1)!k!} + \frac{(n-1)!k}{(n-k)!(k-1)!k} =$$

$$\frac{(n-1)!(n-k)}{(n-k)!k!} + \frac{(n-1)!k}{(n-k)!k!} = \frac{(n-1)!(n-k+k)}{(n-k)!k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

<u>A lesson to remember:</u> we can use algebra in combinatorics.

Theorem.

For every two natural numbers n,k, $k \leq n$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof 2. (King method + induction, outline)

We designate one element of our n-element set X king. k-subsets of X can be partitioned into those who do contain the king and those who don't. Obviously, there are $\binom{n-1}{k}$ k-element subsets who don't contain the king and $\binom{n-1}{k-1}$ who do. Hence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. QED

<u>A lesson to remember</u>: we can use combinatorics to prove algebraic formulae.

Theorem. (Newton's binomial theorem revisited)
For every two numbers a, b

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof 2. (Induction on n)
Trivial for n=1 and n=2. Suppose it holds for some n and consider $(a+b)^{n+1} = (a+b)\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} =$

$$a\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} + b\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} =$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} =$$

$$\sum_{k=1}^{n+1} \binom{n}{k-1} a^{k-1+1} b^{n-(k-1)} + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{k-1} b^{n+1-(k-1)}$$

$$=\sum_{k=1}^{n+1}\binom{n}{k-1}a^{k-1+1}b^{n-(k-1)}+\sum_{k=1}^{n+1}\binom{n}{k-1}a^{k-1}b^{n+1-(k-1)}$$

$$=\sum_{k=1}^{n+1}\binom{n}{k-1}a^{k}b^{n+1-k}+\sum_{k=1}^{n+1}\binom{n}{k-1}a^{k-1}b^{n-k+2}$$

The coefficient of a^0b^{n+1} is $\binom{n}{0}=1=\binom{n+1}{0}$ (k=1 in the second sum)
The coefficient of $a^{n+1}b^0$ is $\binom{n}{n}=1=\binom{n+1}{n+1}$ (k=n+1 in the first sum).
Apart from these two cases, the coefficient of a^pb^{n+1-p} in the first sum is $\binom{n}{p-1}$ and $\binom{n}{p}$ in the second one. Combining these we get the coefficient of a^p equal to $\left(\binom{n}{p-1}+\binom{n}{p}\right)=\binom{n+1}{p}b^{n-p+1}$ thanks to the last theorem. QED

No combinatorics, pure algebra. A nightmare!